Analysis of a High Order Finite Volume Scheme for the Vlasov-Poisson System

Roland Duclous, Bruno Dubroca, Francis Filbet

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ROLAND DUCLOUS, BRUNO DUBROCA AND FRANCIS FILBET

Abstract. We propose a second order finite volume scheme to discretize the one-dimensional Vlasov-Poisson system with boundary conditions. For this problem, a rather general initial and boundary data lead to a unique solution with bounded variations but such a solution becomes discontinuous when the external voltage is large enough. We prove that the numerical approximation converges to the weak solution and show the efficiency of the scheme to simulate beam propagation with several particle species.

Keywords. Finite Volume Schemes, Vlasov-Poisson System, Weak BV Estimate.

AMS subject classifications. 65M12, 82D10

Contents

1. Introduction

The Vlasov-Poisson system is a model for the motion of a collisionless plasma and describes the evolution of the distribution function of particles, solution of the Vlasov equation, under the effects of free transport and self-consistent electric fields given by the Poisson equation. Here, we consider a dilute electron gas emitted at position $x = 0$ and absorbed at $x = L$. It gives rise to a nonlinear system of equations with boundary conditions. Under an external voltage, the dynamics of such a problem is modeled by the following system [15, 17]

$$(1)\qquad \begin{cases} \dfrac{\partial f}{\partial t} + v\,\dfrac{\partial f}{\partial x} + E(t,x)\dfrac{\partial f}{\partial v} = 0, \quad t \geq 0,\ (x,v) \in Q\,; \\[2ex] -\dfrac{\partial^2 \phi}{\partial x^2}(t,x) = \rho(t,x), \quad E(t,x) = -\dfrac{\partial \phi}{\partial x}(t,x); \quad t \geq 0,\ x \in \Omega\,; \\[2ex] f(0,x,v) = f_0(x,v), \quad (x,v) \in Q\,; \end{cases}$$

where $Q := \Omega \times \mathbb{R}$ with $\Omega := (0, L)$. We define the macroscopic charged density $\rho(t,x)$ and the related current density $j(t,x)$ by

$$\rho(t,x) = \int_{-\infty}^{\infty} f(t,x,v)dv, \quad j(t,x) = \int_{-\infty}^{\infty} vf(t,x,v)dv, \quad (t,x) \in \mathbb{R}^+ \times \Omega. \tag{2}$$

Here, the boundary conditions for the electron distribution $f(t,x,v) \geq 0$ are given at $x = 0$

$$f(t,0,v) \;=\; g(t,v) \geq 0, \quad v \;>\; 0; \tag{3}$$

and at $x = L$

$$f(t,L,v) \;=\; 0, \quad v \;<\; 0; \tag{4}$$

and external voltages are given at $x = 0$ and $x = L$:

$$\phi(t,0) \;=\; 0, \quad \phi(t,L) \;=\; -\lambda(t) \;\leq\; 0, \quad t \;\geq\; 0. \tag{5}$$

Mathematical study of such nonlinear boundary value problem was initiated in the pioneering work of C. Greengard and P.-A. Raviart [15], in which stationary solutions are constructed. A higher dimensional generalization was given in [19] and [8]. On the other hand, for the dynamical problem of plane diode (1)-(5), weak solutions can be constructed as in [7]. Finally, recently Y. Guo *et al.* give a complete existence and uniqueness proof for the present model (1)-(5) [17] and for the Vlasov-Maxwell system [14].

The aim of this paper is to propose a high order finite volume scheme for the one-dimensional Vlasov-Poisson equation over an interval and to analyze its convergence. In one or two dimension, the numerical resolution of the Vlasov equation is often performed using eulerian methods. These methods are strongly inspired by the discretization of conservation laws in fluid mechanics [4, 23]. They consist in a discretization of the phase space (x, v), which is done by following the characteristic curves at each time step and interpolating the value at the origin of the characteristics by polynomial [12, 13]. This interpolation method works well for simple geometries of the physical space but does not seem to be well suited to more complex geometries. We refer to [13, 1] for a review of the literature on this topic and notice that more recently, J.A Carrillo *et al* [5] propose new schemes based on WENO reconstructions, which are particularly well suited and efficient for the study of discontinuities propagation.

Another type of schemes for the Vlasov equation is the finite volume type method (or flux balance method), where the discrete unknowns are averages of the distribution function on volumes paving the phase space. These unknowns are updated by considering incoming and outgoing fluxes leading to mass conservation. A high order scheme of this type was introduced by J.P. Boris and D.L. Book [4] for transport equations and later F. Filbet *et al.* proposed an improved version of this scheme, called the Positive and Flux Conservative method (PFC) [12, 13], which is not only conservative, but also preserves the positivity and the maximum value of the distribution function. The scheme was implemented up to third order accuracy. Let us also mention related papers where the convergence of a numerical scheme for the Vlasov-Poisson system is investigated. On the one hand, J. Schaeffer [20] proves the convergence of a finite difference scheme for the Vlasov-Poisson-Fokker-Planck system : transport terms are approximated by a characteristic method whereas diffusive term are treated by a classical finite difference operator. On the other hand, N. Besse studies the convergence of semi-lagrangian methods for smooth solutions to the Vlasov-Poisson [2] but it seems difficult to adapt this methodology for discontinuous solutions. Thus, F. Filbet performs a convergence analysis and gets error estimates on a finite volume scheme [10, 11] for weak BV solutions allowing discontinuities to occur, but this scheme is only first order and is not enough accurate to get a good approximation of the distribution function. Here, we extend the analysis to second order finite volume schemes and investigate the case where the solution may be discontinuous. More precisely, the purpose of this work is to prove the convergence of a second order finite volume scheme for the dynamic of plane diode model problem in plasma physics, namely, the one-dimensional Vlasov-Poisson system with boundary conditions with respect to the space variable.

We first present a second order upwind finite volume scheme computing the fluxes on the boundary of each cell of the mesh. Thus, from an L^∞ estimate on the velocity moments of f_h, we obtain a bound on the discrete electric field in $W^{1,\infty}$. We next give a weak BV inequality which will be useful for the convergence of the approximation to the weak solution to the Vlasov-Poisson system.

2. Numerical scheme and main results

In order to compute a numerical approximation of the solution of the Vlasov-Poisson system, let us define a Cartesian mesh of the phase space $\mathcal{M}_h$ consisting of cells, denoted by $C_{i,j}$, $i \in I = \{0, \dots, n_x - 1\}$, where n_x is the number of subcells of $(0, L)$ and $j \in \mathbb{Z}$. The mesh $\mathcal{M}_h$ is given by an increasing sequence $(x_{i-1/2})_{i\in\{0,\dots,n_x\}}$ of the interval $(0, L)$ and by a second increasing sequence $(v_{j-1/2})_{j\in\mathbb{Z}}$ of $\mathbb{R}$ for the velocity space.

Let $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ be the physical space step and $\Delta v_j = v_{j+1/2} - v_{j-1/2}$ be the velocity space step. The parameter h indicates

$$h = \max_{i,j}\{\Delta x_i, \Delta v_j\}.$$

We assume the mesh satisfies the following condition : there exists $\alpha \in (0, 1)$ such that for all $h > 0$ and $(i, j) \in I \times \mathbb{Z}$,

$$\alpha\, h \le \Delta x_i \le h, \qquad \text{and} \qquad \alpha\, h \le \Delta v_j \le h. \tag{6}$$

Finally, we obtain a Cartesian mesh of the phase space constituted of control volumes

$$C_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (v_{j-1/2}, v_{j+1/2}) \quad \text{for } i \in I \text{ and } j \in \mathbb{Z}.$$

In order to work with a bounded domain, we will truncate at $|v| = v_h$ (v_h sufficiently large which will go to $+\infty$ as $h \to 0$) and we denote by J the following set

$$J := \{j \in \mathbb{Z}, \quad |v_j| \le v_h\}.$$

Let Δt be the time step and $t^n = n\,\Delta t$ and x_i (resp. v_j) represents the middle of $[x_{i-1/2}, x_{i+1/2}]$ (resp. $[v_{j-1/2}, v_{j+1/2}]$). We set the discrete initial datum as

$$f^0_{i,j} = \frac{1}{\Delta x_i\, \Delta v_j} \int_{C_{i,j}} f_0(x, v)\, dx dv.$$

For $n \ge 0$, we define a sequence $(f^{n+1}_{i,j})_{i,j}$, which is assumed to approximate the average of the Vlasov equation solution (1)-(5) on the control volume $C_{i,j}$. It is given by

$$f^{n+1}_{i,j} \;=\; f^n_{i,j} \;-\; \frac{\Delta t}{\Delta x_i}\,[\mathcal{F}_{i+1/2,j} - \mathcal{F}_{i-1/2,j}] \;-\; \frac{\Delta t}{\Delta v_j}\,[\mathcal{G}_{i,j+1/2} - \mathcal{G}_{i,j-1/2}], \tag{7}$$

with

$$\begin{cases} \mathcal{F}_{i+1/2,j} &=& v_j^+\, f^l_{i+1/2,j} \;-\; v_j^-\, f^r_{i+1/2,j}, \\[2mm] \mathcal{G}_{i,j+1/2} &=& E_i^{n+}\, f^l_{i,j+1/2} \;-\; E_i^{n-}\, f^r_{i,j+1/2}, \end{cases} \tag{8}$$

where $f^l_{i+1/2,j}$ and $f^r_{i+1/2,j}$ are second order reconstructions with respect to the space variable $x \in (0, L)$ of the distribution function

$$\begin{cases} f^l_{i+1/2,j} = f^n_{i,j} \;+\; \sigma_{i+1/2,j}\, \left[f^n_{i+1,j} \;-\; f^n_{i,j}\right], \\[2mm] f^r_{i+1/2,j} = f^n_{i+1,j} \;-\; \sigma_{i+3/2,j}\, \left[f^n_{i+2,j} \;-\; f^n_{i+1,j}\right], \end{cases} \tag{9}$$

with the slope $\sigma_{i+1/2,j}$ given by the minmod limiter

$$\sigma_{i+1/2,j} = \begin{cases} 0, & \text{if } (f^n_{i+1,j} - f^n_{i,j})\,(f^n_{i,j} - f^n_{i-1,j}) \leq 0, \\ \dfrac{|f^n_{i,j} - f^n_{i-1,j}|}{|f^n_{i+1,j} - f^n_{i,j}|}\,\dfrac{\Delta x_i}{\Delta x_i + \Delta x_{i-1}} & \text{if } \dfrac{|f^n_{i,j} - f^n_{i-1,j}|}{\Delta x_i + \Delta x_{i-1}} \leq \dfrac{|f^n_{i+1,j} - f^n_{i,j}|}{\Delta x_i + \Delta x_{i+1}}, \\ \dfrac{\Delta x_i}{\Delta x_i + \Delta x_{i+1}}, & \text{else.} \end{cases} \tag{10}$$

Also $f^l_{i,j+1/2}$ and $f^r_{i,j+1/2}$ are built using the same type of reconstruction with respect to the velocity space $v \in \mathbb{R}$ for the flux $\mathcal{G}_{i,j+1/2}$

$$\begin{cases} f^l_{i,j+1/2} = f^n_{i,j} \;+\; \sigma_{i,j+1/2,j}\left[f^n_{i,j+1} \;-\; f^n_{i,j}\right], \\ f^r_{i+1/2,j} = f^n_{i,j+1} \;-\; \sigma_{i,j+3/2}\left[f^n_{i,j+2} \;-\; f^n_{i,j+1}\right], \end{cases} \tag{11}$$

with the slope $\sigma_{i,j+1/2}$ given by the minmod limiter

$$\sigma_{i,j+1/2} = \begin{cases} 0, & \text{if } (f^n_{i,j+1} - f^n_{i,j})\,(f^n_{i,j} - f^n_{i,j-1}) \leq 0, \\ \dfrac{|f^n_{i,j} - f^n_{i,j-1}|}{|f^n_{i,j+1} - f^n_{i,j}|}\,\dfrac{\Delta v_j}{\Delta v_j + \Delta v_{j-1}} & \text{if } \dfrac{|f^n_{i,j} - f^n_{i,j-1}|}{\Delta v_j + \Delta v_{j-1}} \leq \dfrac{|f^n_{i,j+1} - f^n_{i,j}|}{\Delta v_{j+1} + \Delta v_j}, \\ \dfrac{\Delta v_j}{\Delta v_j + \Delta v_{j+1}},, & \text{else.} \end{cases} \tag{12}$$

Let us notice that in this paper, we only consider the case of minmod limiters but we can easily apply the present analysis to classical limiters as superbee, etc. These conditions are sufficent to compute some approximations but we add some limiters useful to prove an error estimate result : there exist $K_1, K_2 > 0$ and $\beta \in (1,2)$ such that

$$\sigma_{i+1/2,j}(f^n_{i,j} - f^n_{i-1,j})^2 \;+\; \sigma_{i,j+1/2}(f^n_{i,j} - f^n_{i,j-1})^2 \;\;\leq\;\; K\,h^\beta, \qquad \forall (i,j) \in I \times \mathbb{Z}. \tag{13}$$

This conditions are used in section 3.3 only, for the consistency result.
The value E^n_i is an approximation of the electric field on $[x_{i-1/2}, x_{i+1/2}]$ given below by computing an approximate solution of the Poisson equation. To complete the scheme for the approximation of the Vlasov equation, we impose boundary conditions on x. To do this, we define two approximations $f^n_{-1,j}$ and $f^n_{n_x,j}$ on “virtual cells”, given by

$$\begin{cases} f^n_{-1,j} \;=\; g^n_j \;:=\; g(t^n, v_j), & \text{if } v_j > 0,\; j \in J, \\ f^n_{n_x,j} \;=\; 0, & \text{if } v_j < 0,\; j \in J, \end{cases} \tag{14}$$

and to define slope limiters in the neighborhood of the boundary we also impose zero slope condition, that is, $f_{-2,j} = f_{-1,j}$ and $f_{n_x+1,j} = f_{n_x,j}$. We also set

$$\mathcal{G}_{i,j+1/2} \;=\; 0, \quad \text{for} \quad (i,j) \;\in\; I \times \mathbb{Z} \setminus J.$$

Thus, we are able to define the numerical solution approximating the solution of the Vlasov equation on $Q_T := \Omega_T \times \mathbb{R}$ by

$$f_h(t,x,v) = \begin{cases} f^n_{i,j}, & \text{if } (t,x,v) \in [t^n, t^{n+1}) \times C_{i,j} \text{ and } (i,j) \in I \times J, \\ 0, & \text{if } |v| > v_h. \end{cases}$$

Computing moments in v of the distribution function f_h, we define the discrete charge and current densities for $(t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$:

$$\begin{aligned}\rho_h(t,x) &= \int_{\mathbb{R}} f_h(t,x,v)dv = \sum_{j\in\mathbb{Z}} \Delta v_j\, f^n_{i,j} = \rho^n_i,\\ j_h(t,x) &= \int_{\mathbb{R}} v\, f_h(t,x,v)dv = \sum_{j\in\mathbb{Z}} \Delta v_j\, v_j\, f^n_{i,j} = j^n_i.\end{aligned}$$

Now, to complete the scheme we apply a finite volume scheme to the electric field's equation. Let us denote E^n_i an approximation of the electric field in $(x_{i-1/2}, x_{i+1/2})$ given by

$$E^n_{i+1} - E^n_i = \Delta x_i\, \rho^n_i, \quad \text{for} \quad i = 0, \ldots, n_x - 2, \tag{15}$$

and is supplemented by the following condition, which comes from the discrete potential

$$\sum_{i=0}^{n_x-1} \Delta x_i\, E^n_i = \lambda(t^n) - 0 = \lambda(t^n). \tag{16}$$

We compute a continuous approximation of the discrete field such that

$$\begin{cases} E_h(t^n, x_i) = E^n_i,\\ \\ E_h \in Q_1([t^n, t^{n+1}] \times [x_{i-1/2}, x_{i+1/2}]),\end{cases} \tag{17}$$

where $Q_1([t^n, t^{n+1}] \times [x_{i-1/2}, x_{i+1/2}])$ represents the space of polynomial of degree one in $[t^n, t^{n+1}] \times [x_{i-1/2}, x_{i+1/2}]$ such that $E_h \in W^{1,\infty}(\Omega_T)$ and $\tilde{E}_h$ is a piecewise constant approximation given by

$$\tilde{E}_h(t,x) = E^n_i, \quad \text{for} \quad (t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2}).$$

We introduce the space

$$BV(Q) = \{u \in L^1(Q), \quad TV_Q(u) < \infty\}$$

where

$$TV_Q(u) = \sup\left\{\int_Q u(x,v)\, \mathrm{div}_{(\mathrm{x,v})}\varphi(x,v)dxdv, \quad \varphi \in C^\infty_c(\overline{Q}), \quad |\varphi(x,v)| \le 1, \quad \forall (x,v) \in Q\right\}.$$

We shall now prove the following theorem of convergence for the numerical approximation.

Theorem 2.1. *Assume for some $p > 2$, $f_0(x,v)$ and $g(t,v)$ satisfy: for all $T > 0$*

$$TV_Q(f_0) + \int_0^T \int_{v\ge 0} [1+v]g(s,v)dvds + \||v|^p f_0\|_\infty + \||v|^p g\|_\infty < \infty. \tag{18}$$

Let $\mathcal{M}_h$ be a Cartesian mesh of the phase space and Δt be the time step satisfying the CFL condition : there exists $\xi \in (0,1)$ such that

$$\frac{3\,\Delta t}{2}\left(\frac{|v_j|}{\Delta x_i} + \frac{C_\lambda}{\Delta v_j}\right) \le 1 - \xi \quad \forall (i,j) \in I \times J, \tag{19}$$

with

$$C_\lambda = \frac{\|\lambda\|_{L^\infty}}{L} + 2\left(\|f^0\|_{L^1} + \int_0^T \int_{v\ge 0} v\, g_h(t,v)dvdt\right).$$

We consider the numerical solution given by the scheme (7)-(12), denoted by $f_h(t,x,v)$, and the discrete self-consistent field $E_h(t,x)$ given by (15)-(17). Then we have

$$\begin{aligned} f_h(t,x,v) &\rightharpoonup f(t,x,v) \text{ weak-}\star \text{ in } L^\infty(Q_T) \quad \text{as } h \to 0,\\ E_h(t,x) &\to E(t,x) \text{ in } C(\overline{\Omega}_T) \quad \text{as } h \to 0,\end{aligned}$$

where (f, E) is the unique solution to the Vlasov-Poisson system (1)–(5), that is for all test functions which belong to

$$\mathcal{T} := \{\varphi \in C_c^1([0,\infty) \times (0,L) \times \mathbb{R}), \quad \varphi(t,0,v) = \varphi(t,L,-v) = 0, \quad \forall v \leq 0\},$$

we have

$$\int_{Q_T} f(t,x,v) \left[\frac{\partial \varphi}{\partial t} + v\, \frac{\partial \varphi}{\partial x}(t,x,v) + E(t,x)\, \frac{\partial \varphi}{\partial v}(t,x,v)\right] dxdvdt + \int_{\Omega_T} f_0(x,v)\varphi(0,x,v)\, dv\, dx + \int_0^T \int_{v\geq 0} v\, [g(t,v)\varphi(t,0,v)]\, dv\, dt = 0$$

and for the electric field

$$\frac{\partial E}{\partial x} = \rho(t,x), \quad \forall (t,x) \in [0,T] \times \Omega,$$

supplemented with boundary conditions.

3. A priori estimates

In this section, we will give some properties satisfied by the numerical approximation as well as by the solution of the continuous problem. We will first prove some properties on the discrete distribution function f_h. From these estimates, we will also give an L^∞ estimate on the electric field E_h. Then, in Proposition 3.2, we will obtain a uniform bound on $|v|^p\, f_h$ in order to obtain an L^∞ estimate on the moments in velocity of f_h and finally a $W^{1,\infty}$ estimate on the discrete electric field E_h.

3.1. **Basic estimates.**

Proposition 3.1. *Assume that $f_0(x,v) \geq 0$ and $g(t,v) \geq 0$ satisfy : for all $T > 0$*

$$\int_0^T \int_{v\geq 0} [1+v] g(s,v) dvds + \| f_0\|_{L^\infty} + \|g\|_{L^\infty} + \|f_0\|_{L^1} < +\infty.$$

Let $\mathcal{M}_h$ be a Cartesian mesh of the phase space and Δt be the time step satisfying the CFL condition: there exists $\xi \in (0,1)$ such that for all $k \in \{0,\ldots,n\}$

$$\frac{\Delta t}{\Delta x_i\, \Delta v_j}\big(\Delta v_j\, |v_j| + \Delta x_i\, |E_i^k|\big) \leq 1-\xi \quad \forall (i,j) \in I \times J. \tag{20}$$

Then, we have

(i) the discrete distribution function at time t^{n+1} satisties the following maximum principle

$$0 \leq f_{i,j}^{n+1} \leq \max\left(\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\right); \quad \forall (i,j) \in I \times \mathbb{Z}; \tag{21}$$

(ii) the discrete density function $\rho_h(t^{n+1})$ satisfies

$$0 \leq \sum_{i\in I} \Delta x_i\, \rho_i^{n+1} \leq \|f^0\|_{L^1} + \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}} \Delta t\, \Delta v_j\, v_j^+\, g_j^k\,; \tag{22}$$

(iii) the discrete electric field is bounded

$$|E_i^{n+1}| \leq \frac{\lambda(t^{n+1})}{L} + 2\left(\|f^0\|_{L^1} + \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}} \Delta t\, \Delta v_j\, v_j^+\, g_j^k\right); \tag{23}$$

(iv) the CFL condition (20) at iteration $n+1$ is satisfied.

Proof: We start from the scheme (9)-(10) and first introduce the following limiters : for $(i,j) \in I \times J$

$$\begin{aligned} f^l_{i+1/2,j} &= f^n_{i,j} + \sigma_{i+1/2,j}\,[f^n_{i+1,j} - f^n_{i,j}],\\ &= f^n_{i,j} + s_{i+1/2,j}\,[f^n_{i,j} - f^n_{i-1,j}], \end{aligned}$$

with

$$s_{i+1/2,j} = \begin{cases} 0, & \text{if } (f^n_{i+1,j} - f^n_{i,j})\,(f^n_{i,j} - f^n_{i-1,j}) \le 0,\\[2mm] \dfrac{|f^n_{i+1,j} - f^n_{i,j}|}{|f^n_{i,j} - f^n_{i-1,j}|}\,\dfrac{\Delta x_i}{\Delta x_i + \Delta x_{i+1}} & \text{if } \dfrac{|f^n_{i+1,j} - f^n_{i,j}|}{\Delta x_i + \Delta x_{i+1}} \le \dfrac{|f^n_{i,j} - f^n_{i-1,j}|}{\Delta x_i + \Delta x_{i-1}},\\[2mm] \dfrac{\Delta x_i}{\Delta x_i + \Delta x_{i-1}}, & \text{else} \end{cases} \tag{24}$$

and

$$\begin{aligned} f^r_{i+1/2,j} &= f^n_{i+1,j} - \sigma_{i+3/2,j}\,[f^n_{i+2,j} - f^n_{i+1,j}],\\ &= f^n_{i+1,j} - s_{i+3/2,j}\,[f^n_{i+1,j} - f^n_{i,j}]. \end{aligned}$$

Also, $f^l_{i,j+1/2}$ and $f^r_{i,j+1/2}$ can be re-written in a similar way

$$\begin{aligned} f^l_{i,j+1/2} &= f^n_{i,j} + s_{i,j+1/2}\,[f^n_{i,j} - f^n_{i,j-1}],\\ f^r_{i,j+1/2} &= f^n_{i,j+1} - s_{i,j+3/2}\,[f^n_{i,j+1} - f^n_{i,j}], \end{aligned}$$

with

$$s_{i,j+1/2} = \begin{cases} 0, & \text{if } (f^n_{i,j+1} - f^n_{i,j})\,(f^n_{i,j} - f^n_{i,j-1}) \le 0,\\[2mm] \dfrac{|f^n_{i,j+1} - f^n_{i,j}|}{|f^n_{i,j} - f^n_{i,j-1}|}\,\dfrac{\Delta v_j}{\Delta v_j + \Delta v_{j+1}}, & \text{if } \dfrac{|f^n_{i,j+1} - f^n_{i,j}|}{\Delta v_j + \Delta v_{j+1}} \le \dfrac{|f^n_{i,j} - f^n_{i,j-1}|}{\Delta v_j + \Delta v_{j-1}},\\[2mm] \dfrac{\Delta v_j}{\Delta v_j + \Delta v_{j-1}}, & \text{else,} \end{cases} \tag{25}$$

where we observe that $0 \le s_{i,j+1/2},\, s_{i+1/2,j} < 1$. Using the scheme (7)-(12), we explicitly write the value of the numerical solution at iteration $n+1$, in terms of the values at time t^n in a better way,

$$\begin{aligned} f^{n+1}_{i,j} = f^n_{i,j} &- \frac{v^+_j\,\Delta t}{\Delta x_i}\left[1 + s_{i+1/2,j} - \sigma_{i-1/2,j}\right](f^n_{i,j} - f^n_{i-1,j})\\ &+ \frac{v^-_j\,\Delta t}{\Delta x_i}\left[1 - s_{i+3/2,j} + \sigma_{i+1/2,j}\right](f^n_{i+1,j} - f^n_{i,j})\\ &- \frac{E^{n+}_i\,\Delta t}{\Delta v_j}\left[1 + s_{i,j+1/2} - \sigma_{i,j-1/2}\right](f^n_{i,j} - f^n_{i,j-1})\\ &+ \frac{E^{n-}_i\,\Delta t}{\Delta v_j}\left[1 - s_{i,j+3/2} + \sigma_{i,j+1/2}\right](f^n_{i,j+1} - f^n_{i,j}). \end{aligned}$$

Under the stability condition (20), the discrete distribution function $f^{n+1}_{i,j}$ could be written as a convex combination of $f^n_{i,j}, f^n_{i-1,j}, f^n_{i+1,j}, f^n_{i,j-1}, f^n_{i,j+1}$; it yields the nonnegativity of $f^{n+1}_{i,j}$ for all $(i,j) \in I \times \mathbb{Z}$. Thus we get the result

$$0 \le f^{n+1}_{i,j} \le \max\left(\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\right).$$

Now, we give an estimate of total mass evolution : for $k \in \{0,\dots,n\}$ we multiply (7)-(12) by $\Delta x_i\,\Delta v_j$ and sum over $(i,j) \in I \times \mathbb{Z}$. It gives,

$$\sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{k+1} \,+\, \Delta t \sum_{j\in\mathbb{Z}} \Delta v_j \left[v_j^-\, f^r_{-1/2,j} + v_j^+ f^l_{n_x+1/2,j}\right]$$
$$= \sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{k} \,+\, \Delta t \sum_{j\in\mathbb{Z}} \Delta v_j \left[v_j^+\, f^r_{-1/2,j} + v_j^- f^l_{n_x+1/2,j}\right].$$

Then, using boundary conditions (14), it yields

$$\sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{k+1} \,+\, \Delta t \sum_{j\in\mathbb{Z}} \Delta v_j \left[v_j^-\, f^k_{0,j} + v_j^+ f^k_{n_x-1,j}\right]$$
$$= \sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{k} \,+\, \Delta t \sum_{j\in\mathbb{Z}} \Delta v_j\, v_j^+\, g_j^k$$

and summing over $k \in \{0,\dots,n\}$ we get

$$\sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{n+1} \,+\, \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} \Delta t\,\Delta v_j \left[v_j^-\, f^k_{0,j} + v_j^+\, f^k_{n_x-1,j}\right]$$
$$= \sum_{(i,j)\in I\times\mathbb{Z}} \Delta x_i\,\Delta v_j\, f_{i,j}^{0} \,+\, \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} \Delta t\,\Delta v_j\, v_j^+\, g_j^k,$$

which gives the result

$$\sum_{i\in I} \Delta x_i\, \rho_i^{n+1} \,\leq\, \|f^0\|_{L^1} + \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} \Delta t\,\Delta v_j\, v_j^+\, g_j^k.$$

Now, let us prove that the discrete electric field is bounded at iteration $n+1$. The argument is the same as in the continuous case: using the scheme (15)-(16), we have $E_0^{n+1} = C^{n+1}$ and for $i = \{1,\dots,n_x-1\}$

$$\begin{aligned} |E_i^{n+1}| &= \left| C^{n+1} + \sum_{k=0}^{i-1} \Delta x_k\, \rho_k^{n+1} \right|, \\ &\leq C^{n+1} + \|f^0\|_{L^1} + \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} \Delta t\,\Delta v_j\, v_j^+\, g_j^k, \end{aligned}$$

where C^{n+1} is such that the relation (16) is satisfied at iteration $n+1$, so that

$$C^{n+1} = \frac{\lambda(t^{n+1}) \,+\, \sum_{i\in I} \Delta x_i (L - x_{i-1/2})\, \rho_i^{n+1}}{L},$$

which proves (23)

$$|E_i^{n+1}| \leq \frac{\lambda(t^{n+1})}{L} + 2\left(\|f^0\|_{L^1} + \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} \Delta t\,\Delta v_j\, v_j^+\, g_j^k \right).$$

Finally, from this latter bound we check that the CFL condition (19) is satistied at time $n+1$. $\square$

3.2. **Estimates on the electric field.** Now, let us give a uniform bound on $|v|^p\, f_h$ for $p>2$, which will lead to a uniform bound on the and an estimate on first moments ρ_h and j_h throughout an energy estimate.

Proposition 3.2. *Assume that for $p>2$ and for all $(t,x,v)\in Q_T$*

$$|v|^p\, f_0(x,v) \,+\, |v|^p\, g(t,v) \,<\, \infty$$

and $\|\lambda\|_{W^{1,\infty}}<\infty$. Then, there exists $C_T>0$, ony depending on f_0, g, λ and α, such that

$$0 \,\leq\, \max_{i,j}\{|v_{j-1/2}|^p\, f^n_{i,j}\} \leq C_T. \tag{26}$$

Moreover, there exists $C_T>0$, for all $(n,i)\in\{0,\ldots,N_T\}\times I$,

$$\left|\frac{E_i^{n+1}-E_i^n}{\Delta t}\right| + \left|\frac{E^n_{i+1}-E^n_i}{\Delta x_i}\right| \,\leq\, C_T. \tag{27}$$

Proof: For $p>2$, we multiply the scheme (7)-(12) by $|v_{j-1/2}|^p$ and using the reconstruction proposed in (24)-(25), we have

$$\begin{array}{rcl}
|v_{j-1/2}|^p\, f^{n+1}_{i,j} &=& |v_{j-1/2}|^p\, f^n_{i,j} \\
&-& \dfrac{v_j^+\,\Delta t}{\Delta x_i}\left[1+s_{i+1/2,j}-\sigma_{i-1/2,j}\right]\left(|v_{j-1/2}|^p\, f^n_{i,j}-|v_{j-1/2}|^p\, f^n_{i-1,j}\right)\\
&+& \dfrac{v_j^-\,\Delta t}{\Delta x_i}\left[1-s_{i+3/2,j}+\sigma_{i+1/2,j}\right]\left(|v_{j-1/2}|^p\, f^n_{i+1,j}-|v_{j-1/2}|^p\, f^n_{i,j}\right)\\
&-& \dfrac{E_i^{n+}\,\Delta t}{\Delta v_j}\left[1+s_{i,j+1/2}-\sigma_{i,j-1/2}\right]\left(|v_{j-1/2}|^p\, f^n_{i,j}-|v_{j-3/2}|^p\, f^n_{i,j-1}\right)\\
&+& \dfrac{E_i^{n-}\,\Delta t}{\Delta v_j}\left[1-s_{i,j+3/2}+\sigma_{i,j+1/2}\right]\left(|v_{j+1/2}|^p\, f^n_{i,j+1}-|v_{j-1/2}|^p\, f^n_{i,j}\right).\\
&-& \dfrac{E_i^{n+}\,\Delta t}{\Delta v_j}\left[1+s_{i,j+1/2}-\sigma_{i,j-1/2}\right]\left(|v_{j-3/2}|^p\,-\,|v_{j-1/2}|^p\right) f^n_{i,j-1}\\
&+& \dfrac{E_i^{n-}\,\Delta t}{\Delta v_j}\left[1-s_{i,j+3/2}+\sigma_{i,j+1/2}\right]\left(|v_{j-1/2}|^p\,-|v_{j+1/2}|^p\right) f^n_{i,j+1}.
\end{array}$$

Then, using that $\left||v_{j+1/2}|^p-|v_{j-1/2}|^p\right| \,\leq\, p\,(1+|v_{j-1/2}|^p)\,\Delta v_j$ and from the CFL condition (20), we get

$$\begin{array}{rcl}
\max\limits_{i,j}\{|v_{j-1/2}|^p\, f^{n+1}_{i,j}\} &\leq& \max\limits_{i,j}\{|v_{j-1/2}|^p\, f^n_{i,j}\}\\
&+& \Delta t\,\dfrac{3\,p\,\|E_h\|_{L^\infty}}{2\,\alpha}\left(\max\limits_{i,j}\{|v_{j-1/2}|^p\, f^n_{i,j}\} \,+\, \|f_h\|_{L^\infty}\right).
\end{array}$$

It finally yields using a discrete version of Gronwall's lemma and taking into account boundary conditions

$$\max_{i,j}\{|v_{j-1/2}|^p\, f^n_{i,j}\} \leq \left(\max_{i,j}\{|v_{j-1/2}|^p\,[g^n_j+f^0_{i,j}]\}+\|f_h(0)\|_{L^\infty}+\|g_h\|_{L^\infty}\right)\exp\left(\frac{3\,p\,\|E_h\|_{L^\infty}}{2\,\alpha}\,t^n\right).$$

We remind that in Proposition 3.1, we have already seen that E_h is bounded in L^∞. On the one hand from the latter estimate, we can prove a uniform upper bound on the discrete density

$$\rho^n_i=\sum_{j\in\mathbb{Z}}\Delta v_j\, f^n_{i,j}\leq \|f_h(t^n)\|_{L^\infty} \,+\, \max_{i,j}\{|v_{j-1/2}|^p\, f^n_{i,j}\}\sum_{|v_{j-1/2}|\geq 1}\frac{\Delta v_j}{|v_{j-1/2}|^p}\leq C_T.$$

Therefore, from the finite volume scheme for E_i^n we get

$$\left|\frac{E_{i+1}^n - E_i^n}{\Delta x_i}\right| = \rho_i^n \leq C_T.$$

On the other hand, we give a uniform upper bound on the jump

$$\left|\frac{E_i^{n+1} - E_i^n}{\Delta t}\right|.$$

Using the finite volume scheme for E_i^n (15)-(16) and the scheme for the distribution function $f_{i,j}^n$ (7)-(12), we get a new formulation

$$\begin{aligned}\frac{E_i^{n+1} - E_i^n}{\Delta t} &= \frac{C^{n+1} - C^n}{\Delta t} + \sum_{k=0}^{i} \frac{\rho_k^{n+1} - \rho_k^n}{\Delta t} \\ &= \frac{C^{n+1} - C^n}{\Delta t} - j_{i+1/2}^n + j_{-1/2}^n,\end{aligned}$$

with

$$j_{i+1/2}^n = \sum_{j\in\mathbb{Z}} \Delta v_j \, [v_j^+ f_{i+1/2,j}^l - v_j^- f_{i+1/2,j}^r],$$

or

$$\frac{E_i^{n+1} - E_i^n}{\Delta t} = \frac{\lambda(t^{n+1}) - \lambda(t^n)}{L\,\Delta t} - j_{i+1/2}^n - \frac{1}{L}\sum_{k\in I} \Delta x_k \, j_{k+1/2}^n.$$

It remains to get an upper bound of $j_{i+1/2}^n$, which can be done from (26). We have for $h \leq 1$

$$\begin{aligned}|j_{i+1/2}^n| &\leq \sum_{j\in\mathbb{Z}} \Delta v_j \, [v_j^+ f_{i+1/2,j}^l + v_j^- f_{i+1/2,j}^r], \\ &\leq 4\sum_{j\in\mathbb{Z}} \Delta v_j \left(1 + |v_{j-1/2}|\right) f_{i,j}^n \leq C_T.\end{aligned}$$

Thus under the assumption $\lambda \in W^{1,\infty}(0,T)$, it yields

$$\left|\frac{E_i^{n+1} - E_i^n}{\Delta t}\right| \leq C_T.$$

□

3.3. **Weak BV estimate for f_h.** The following lemma will be useful to obtain the convergence of (E_h, f_h) to the Vlasov equation solution.

Lemma 3.3. *Under the stability condition (20) on the time step and the condition on the mesh (6), assume the initial data belong to $L^1(Q) \cap L^\infty(Q)$. Consider $R > 0$ and $T > 0$ with $h < R$ and $\Delta t < T$. Let $j_0, j_1 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ be such that $-R \in (v_{j_0-1/2}, v_{j_0+1/2})$, $R \in (v_{j_1-1/2}, v_{j_1+1/2})$, and $T \in ((N_T-1)\Delta t, N_T\,\Delta t)$. We define*

$$\begin{aligned}EF_{1h} = \Delta t \sum_{n=0}^{N_T} \sum_{i\in I} \sum_{j=j_0}^{j_1} \Delta x_i \, \Delta v_j \Big[& v_j^+ \, |f_{i,j}^n - f_{i-1,j}^n| + v_j^- \, |f_{i,j}^n - f_{i+1,j}^n| \\ & + E_i^{n+} \, |f_{i,j}^n - f_{i,j-1}^n| + E_i^{n-} \, |f_{i,j}^n - f_{i,j+1}^n| \Big] \end{aligned} \tag{28}$$

and

$$EF_{2h} = \Delta t \sum_{n=0}^{N_T} \sum_{i\in I} \sum_{j=j_0}^{j_1} \Delta x_i \, \Delta v_j \left| f_{i,j}^{n+1} - f_{i,j}^n \right|. \tag{29}$$

Then, there exists $C > 0$ depending only on T, R, f_0, α, ξ such that

$$EF_{1h} \;\leq\; C\, h^{1/2} \quad \textit{and} \quad EF_{2h} \;\leq\; C\, \Delta t^{1/2}. \tag{30}$$

Proof: Multiplying the scheme (7)-(12) by $\Delta x_i \, \Delta v_j \, f_{i,j}^n$ and summing over $i \in \{0, \ldots, n_x - 1\}$, $j \in \{j_0, \ldots, j_1\}$, and $n \in \{0, \ldots, N_T\}$, it follows that

$$B_1 \,+\, B_2 \,=\, 0,$$

where

$$B_1 \;=\; \sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n]\, f_{i,j}^n.$$

$$B_2 \;=\; \Delta t \sum_{n,i,j} \Delta v_j \, [\mathcal{F}_{i+1/2,j} - \mathcal{F}_{i-1/2,j}]\, f_{i,j}^n \;+\; \Delta x_i \, [\mathcal{G}_{i,j+1/2} - \mathcal{G}_{i,j-1/2}]\, f_{i,j}^n$$

Noting that

$$[f_{i,j}^{n+1} - f_{i,j}^n]\, f_{i,j}^n = -\frac{1}{2}[f_{i,j}^{n+1} - f_{i,j}^n]^2 - \frac{1}{2}(f_{i,j}^n)^2 + \frac{1}{2}(f_{i,j}^{n+1})^2,$$

then

$$B_1 \;=\; -\frac{1}{2} \sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n]^2 \;-\; \frac{1}{2} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^0)^2 + \frac{1}{2} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^{N_T+1})^2.$$

By scheme (7)-(12), we have

$$\begin{aligned}
&\sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n]^2 \\
&\quad = \sum_{n,i,j} \frac{\Delta t^2}{\Delta x_i \Delta v_j} \Big[\Delta v_j \, v_j^+ \, \big(1 + s_{i+1/2,j} - \sigma_{i-1/2,j}\big)\, (f_{i,j}^n - f_{i-1,j}^n) \, + \\
&\qquad\qquad \Delta v_j \, v_j^- \, \big(1 - s_{i+3/2,j} + \sigma_{i+1/2,j}\big)\, (f_{i,j}^n - f_{i+1,j}^n) \, + \\
&\qquad\qquad \Delta x_i \, E_i^{n+} \, \big(1 + s_{i,j+1/2} - \sigma_{i,j-1/2}\big)\, (f_{i,j}^n - f_{i,j-1}^n) \, + \\
&\qquad\qquad \Delta x_i \, E_i^{n-} \, \big(1 - s_{i,j+3/2} + \sigma_{i,j+1/2}\big)\, (f_{i,j}^n - f_{i,j+1}^n) \Big]^2 .
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the stability condition (20), there exists $C_0 > 0$, only depending on f_0, such that

$$\begin{aligned}
B_1 \;\geq\; & -\frac{\Delta t}{2} (1 - \xi) \sum_{n,i,j} \Big[\Delta v_j \, v_j^+ \, \big(1 + s_{i+1/2,j} - \sigma_{i-1/2,j}\big)\, (f_{i,j}^n - f_{i-1,j}^n)^2 \\
& \quad + \Delta v_j \, v_j^- \, \big(1 - s_{i+3/2,j} + \sigma_{i+1/2,j}\big)\, (f_{i,j}^n - f_{i+1,j}^n)^2 \\
& \quad + \Delta x_i \, E_i^{n+} \, \big(1 + s_{i,j+1/2} - \sigma_{i,j-1/2}\big)\, (f_{i,j}^n - f_{i,j-1}^n)^2 \\
& \quad + \Delta x_i \, E_i^{n-} \, \big(1 - s_{i,j+3/2} + \sigma_{i,j+1/2}\big)\, (f_{i,j}^n - f_{i,j+1}^n)^2 \Big] \;-\; C_0.
\end{aligned}$$

Now, we study the term B_2, which may be rewritten as $B_2 = B_{21}+B_{22}$ where B_{21} is the contribution of the first order approximation

$$\begin{aligned} B_{21} &= \frac{\Delta t}{2}\sum_{n,i,j}\Big[\Delta v_j\, v_j^+\,[f^n_{i,j}-f^n_{i-1,j}]^2 + \Delta v_j\, v_j^-\,[f^n_{i,j}-f^n_{i+1,j}]^2 + \\ &\qquad \Delta x_i\, E_i^{n+}\,[f^n_{i,j}-f^n_{i,j-1}]^2 + \Delta x_i\, E_i^{n-}\,[f^n_{i,j}-f^n_{i,j+1}]^2\Big] \\ &+ \frac{\Delta t}{2}\sum_{n,i}\Big[\Delta x_i\, E_i^{n+}\,[(f^n_{i,j_1})^2-(f^n_{i,j_0-1})^2] + \Delta x_i\, E_i^{n-}\,[(f^n_{i,j_0})^2-(f^n_{i,j_1+1})^2]\Big] \\ &+ \frac{\Delta t}{2}\sum_{n,j}\Big[\Delta v_j\, v_j^+\,[(f^n_{i_1,j})^2-(f^n_{i_0-1,j})^2] + \Delta v_j\, v_j^-\,[(f^n_{i_0,,j})^2-(f^n_{i_1+1,j})^2]\Big] \end{aligned}$$

and B_{22} the contribution of the second order term

$$\begin{aligned} B_{22} &= \Delta t\sum_{n,i,j}\Delta v_j\left[v_j^+[s_{i+1/2,j}-\sigma_{i-1/2,j}][f^n_{i,j}-f^n_{i-1,j}] + v_j^-[s_{i+3/2,j}-\sigma_{i+1/2,j}]\,[f^n_{i+1,j}-f^n_{i,j}]\right] f^n_{i,j} + \\ &\qquad \Delta x_i\left[E_i^{n+}\,[s_{i,j+1/2}-\sigma_{i,j-1/2}]\,[f^n_{i,j}-f^n_{i,j-1}] + E_i^{n-}\,[s_{i,j+3/2}-\sigma_{i,j+1/2}]\,[f^n_{i,j+1}-f^n_{i,j}]\right] f^n_{i,j}. \end{aligned}$$

On the one hand, from the estimates on velocity moments in Proposition 3.2, we get that there exists a constant $C_1 > 0$, only depending on T and f_0, such that

$$\begin{aligned} B_{21} &\geq \frac{\Delta t}{2}\sum_{n,i,j}\Big[\Delta v_j\, v_j^+\,[f^n_{i,j}-f^n_{i-1,j}]^2 + \Delta v_j\, v_j^-\,[f^n_{i,j}-f^n_{i+1,j}]^2 + \\ &\qquad \Delta x_i\, E_i^{n+}\,[f^n_{i,j}-f^n_{i,j-1}]^2 + \Delta x_i\, E_i^{n-}\,[f^n_{i,j}-f^n_{i,j+1}]^2\Big] - C_1. \end{aligned}$$

On the other hand, using that

$$s_{i+1/2,j}(f^n_{i,j}-f^n_{i-1,j}) = \sigma_{i+1/2,j}(f^n_{i+1,j}-f^n_{i,j}),$$

and

$$s_{i,j+1/2}(f^n_{i,j}-f^n_{i,j-1}) = \sigma_{i,j+1/2}(f^n_{i,j+1}-f^n_{i,j}),$$

we prove that there exists a constant C_2, only depending on T and f_0, such that

$$\begin{aligned} B_{22} &\geq -\Delta t\sum_{n,i,j}\Big[\Delta v_j\,\left(v_j^+\,\sigma_{i-1/2,j}\,[f^n_{i,j}-f^n_{i-1,j}]^2 + v_j^-\,s_{i+3/2,j}\,[f^n_{i,j}-f^n_{i+1,j}]^2\right) + \\ &\qquad \Delta x_i\,\left(E_i^{n+}\,\sigma_{i,j-1/2}\,[f^n_{i,j}-f^n_{i,j-1}]^2 + E_i^{n-}\,s_{i,j+3/2}\,[f^n_{i,j}-f^n_{i,j+1}]^2\right)\Big] - C_2. \end{aligned}$$

Then, since $B_1 + B_{21} + B_{22} = 0$ the following inequality holds:

$$\begin{aligned} &\frac{\xi\,\Delta t}{2}\sum_{n,i,j}\Big[\Delta v_j\, v_j^+\,[f^n_{i,j}-f^n_{i-1,j}]^2 + \Delta v_j\, v_j^-\,[f^n_{i,j}-f^n_{i+1,j}]^2 \\ &\qquad + \Delta x_i\, E_i^{n+}\,[f^n_{i,j}-f^n_{i,j-1}]^2 + \Delta x_i\, E_i^{n-}\,[f^n_{i,j}-f^n_{i,j+1}]^2\Big] \\ \leq\ & \Delta t\sum_{n,i,j}\Delta v_j\,|v_j|\,[s_{i+1/2,j}+\sigma_{i-1/2,j}]\,[f^n_{i,j}-f^n_{i-1,j}]^2 + \Delta x_i\,|E_i^n|\,[s_{i,j+1/2}+\sigma_{i,j-1/2}]\,[f^n_{i,j}-f^n_{i,j-1}]^2 \\ & + C_0 + C_1 + C_2. \end{aligned}$$

Therefore, using hypothesis on the limiters (13), there exists a constant $C > 0$, only depending on f_0, T, R and ξ, such that

$$\begin{aligned}
&\frac{\Delta t}{2} \sum_{n,i,j} \Bigg[\Delta v_j \, v_j^+ \, [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j \, v_j^- \, [f_{i,j}^n - f_{i+1,j}^n]^2 + \\
&\qquad \Delta x_i \, E_i^{n+} \, [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i \, E_i^{n-} \, [f_{i,j}^n - f_{i,j+1}^n]^2 \Bigg], \\
&\le \frac{C}{\xi} \left(1 + K \, h^{\beta-1} \right).
\end{aligned}$$

Finally, the previous inequality and the Cauchy–Schwarz inequality lead to

$$\begin{aligned}
EF_{1h} \; \le \; & \Bigg[\Delta t \sum_{n,i,j} \Delta v_j \, v_j^+ \, [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j \, v_j^- \, [f_{i,j}^n - f_{i+1,j}^n]^2 \\
& + \Delta x_i \, E_i^{n+} \, [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i \, E_i^{n-} \, [f_{i,j}^n - f_{i,j+1}^n]^2 \Bigg]^{1/2} \\
& \times \Bigg[\Delta t \sum_{n,i,j} \Delta x_i^2 (\Delta v_j \, |v_j| + \Delta x_i \, |E_i^n|) \Bigg]^{1/2}, \\
\le \; & h^{1/2} \left(\frac{C}{\xi} \left(1 + K \, h^{\beta-1} \right) \right)^{1/2} \Big[2 \, T \, L \, R \, (1-\xi) \Big]^{1/2}.
\end{aligned}$$

Now, we prove the second estimate on EF_{2h}, using the scheme (7)-(12):

$$\begin{aligned}
EF_{2h} \; = \; & \Delta t \sum_{n,i,j} \Delta x_i \Delta v_j \left| f_{i,j}^{n+1} - f_{i,j}^n \right|, \\
\le \; & \Delta t^2 \sum_{n,i,j} \Bigg[\Delta v_j \, v_j^+ \, |f_{i,j}^n - f_{i-1,j}^n| + \Delta v_j \, v_j^- \, |f_{i,j}^n - f_{i+1,j}^n| \\
& + \Delta x_i \, E_i^{n+} \, |f_{i,j}^n - f_{i,j-1}^n| + \Delta x_i \, E_i^{n-} \, |f_{i,j}^n - f_{i,j+1}^n| \Bigg].
\end{aligned}$$

As in the previous case, we use the Cauchy–Schwarz inequality and the stability condition (20). We also recall that the discrete electric field is uniformly bounded:

$$EF_{2h} \le \Delta t^{1/2} \left[2 \, T \, L \, R \, (1-\xi) \, \frac{C}{\xi} \, \left(1 + K \, h^{\beta-1} \right) \right]^{1/2}.$$

□

4. Proof of Theorem 2.1

In a first part, we prove that there are subsequences which converge to a limit (f, E) and in a second step we identify this limit as the unique solution to the Vlasov-Poisson system (1)-(5).

4.1. **Compactness of the sequence** (f_h, E_h)**.** We consider a sequence of a mesh of the phase space defined as in the beginning of the paper satisfying the condition (6), and we define a time step Δt such that the stability condition (20) is true. This sequence is denoted by $(\mathcal{M}_h)_{h>0}$.

For a given mesh, we are able to construct, by the finite volume scheme (7)-(12), a unique pair (f_h, E_h). Thus, we set

$$A = \Big\{ E_h \in W^{1,\infty}(\Omega_T)\,; \quad E_h \text{ given by (17) for a mesh } \mathcal{M}_h \Big\}.$$

On the one hand, in Proposition 3.1 and Proposition 3.2 we have proved there exists a constant independent on the mesh $\mathcal{M}_h$ such that

$$\|E_h\|_{L^\infty} + \left\| \frac{\partial E_h}{\partial t} \right\|_{L^\infty} + \left\| \frac{\partial E_h}{\partial x} \right\|_{L^\infty} \leq C_T.$$

Moreover, from the same estimates, we also have

$$\|E_h - \tilde{E}_h\|_{L^\infty} \leq C_T\,(h + \Delta t)$$

On the other hand, using the fact that the injection from $W^{1,\infty}(\Omega_T)$ to $C^0(\overline{\Omega}_T)$ is compact, there exists a subsequence of $(E_h)_{h>0}$ and a function E belonging to $C^0(\overline{\Omega}_T)$ such that

$$E_h \rightharpoonup E \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as} \quad h \to 0,$$

and

$$E_h \to E \quad \text{in } C^0(\overline{\Omega}_T) \text{ strong} \quad \text{as} \quad h \to 0\,;$$
$$\tilde{E}_h \to E \quad \text{in } C^0(\overline{\Omega}_T) \text{ strong} \quad \text{as} \quad h \to 0.$$

Moreover, we also know by Proposition 3.1 that the discrete distribution function f_h is bounded in $L^\infty(Q_T)$. Therefore, there exists a subsequence and a function $f \in L^\infty(Q_T)$ such that

$$f_h(t,x,v) \rightharpoonup f(t,x,v) \quad \text{in } L^\infty(Q_T) \text{ weak-}\star \quad \text{as} \quad h \to 0.$$

The discrete charge ρ_h is bounded in $L^\infty(\Omega_T)$; then up to the extraction of a subsequence, we also have

$$\rho_h(t,x) \rightharpoonup \rho(t,x) \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as} \quad h \to 0.$$

4.2. **Convergence to the weak solution of the Vlasov equation.** Let $\varphi \in \mathrm{C}_c^\infty(Q_T)$, $R > 0$, and $j_0, j_1 \in \mathbb{Z}$ be such that

$$\mathrm{Supp}\Big(\varphi(t,x,.)\Big) \subset [-R, R]$$

and

$$-R \in (v_{j_0-1/2}, v_{j_0+1/2}) \quad \text{and} \quad R \in (v_{j_1-1/2}, v_{j_1+1/2}).$$

Moreover $\varphi(t,0,v) = 0$ for all $v \leq 0$ and $\varphi(t,L,v) = 0$ for all $v \geq 0$.

We set $\varphi^n_{i,j}$ such that

$$\varphi^n_{i,j} := \frac{1}{\Delta t\, \Delta x_i\, \Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v)\, dx dv dt$$

and multiply the finite volume scheme (7)-(12) by $\varphi^n_{i,j}$, sum over $i \in \{0, \ldots, n_x - 1\}$, $j \in \{j_0, \ldots, j_1\}$, and $n \in \{0, \ldots, N_T = \frac{T}{\Delta t}\}$,

$$E_1 + E_2 + E_3 = 0,$$

with

$$E_1 = \sum_{n,i,j} (f^{n+1}_{i,j} - f^n_{i,j}) \Delta x_i\, \Delta v_j\, \varphi^n_{i,j},$$

$$\begin{aligned}E_2 &= \sum_{n,i,j}\Bigg[\Delta v_j\, v_j^+\,(f_{i,j}^n - f_{i-1,j}^n) + \Delta v_j\, v_j^-\,(f_{i,j}^n - f_{i+1,j}^n) + \Delta x_i\, E_i^{n+}\,(f_{i,j}^n - f_{i,j-1}^n)\\ &\qquad + \Delta x_i\, E_i^{n-}\,(f_{i,j}^n - f_{i,j+1}^n)\Bigg]\Delta t\,\varphi_{i,j}^n\end{aligned}$$

and

$$\begin{aligned}E_3 &= \sum_{n,i,j}\Bigg[\Delta v_j\, v_j^+\,\left[s_{i+1/2,j} - \sigma_{i-1/2,j}\right]\,(f_{i,j}^n - f_{i-1,j}^n)\\ &\qquad + \Delta v_j\, v_j^-\,\left[\sigma_{i+1/2,j} - s_{i+3/2,j}\right]\,(f_{i,j}^n - f_{i+1,j}^n)\\ &\qquad + \Delta x_i\, E_i^{n+}\,\left[s_{i,j+1/2} - \sigma_{i,j-1/2}\right]\,(f_{i,j}^n - f_{i,j-1}^n)\\ &\qquad + \Delta x_i\, E_i^{n-}\,\left[\sigma_{i,j+1/2} - s_{i,j+3/2}\right]\,(f_{i,j}^n - f_{i,j+1}^n)\Bigg]\Delta t\,\varphi_{i,j}^n.\end{aligned}$$

Moreover, we denote $E_{1,0}$ and $E_{2,0}$ by

$$E_{1,0} = \int_{Q_T} f_h(t,x,v)\frac{\partial\varphi}{\partial t}(t,x,v)dtdxdv \;+\; \int_Q f_0(x,v)\varphi(0,x,v)dxdv$$

and

$$\begin{aligned}E_{2,0} &= \int_{Q_T} f_h(t,x,v)\left[v\,\frac{\partial\varphi}{\partial x}(t,x,v) \;+\; E_h(t,x)\,\frac{\partial\varphi}{\partial v}(t,x,v)\right]dxdvdt\\ &\quad + \int_0^T\int_{v\geq 0} v\,\left[g_h(t,0,v)\varphi(t,0,v)\right]\,dv\,dt.\end{aligned}$$

In the sequel we will compare E_1 with $E_{1,0}$ and E_2 with $E_{2,0}$ to establish that $E_{1,0}+E_{2,0}$ goes to zero as $h\to 0$. We first treat the terms E_1 and $E_{1,0}$ and remark that $E_{1,0}$ can be rewritten as

$$E_{1,0} = \sum_{n,i,j} f_{i,j}^n \int_{C_{i,j}}\left[\varphi(t^{n+1},x,v) - \varphi(t^n,x,v)\right]dx\,dv \;+\; \int_Q f_0(x,v)\,\varphi(0,x,v)\,dx\,dv.$$

By a discrete integration by parts, it follows that

$$\begin{aligned}E_{1,0} &= -\sum_{n,i,j}\left(f_{i,j}^{n+1} - f_{i,j}^n\right)\int_{C_{i,j}}\varphi(t^{n+1},x,v)dxdv\\ &\quad - \int_Q\Big(f_h(0,x,v) - f_0(x,v)\Big)\varphi(0,x,v)dxdv.\end{aligned}$$

Thus,

$$\begin{aligned}|E_1 + E_{1,0}| &\leq \sum_{n,i,j}|f_{i,j}^{n+1} - f_{i,j}^n|\int_{t^n}^{t^{n+1}}\int_{C_{i,j}}\left|\frac{\partial\varphi}{\partial t}(t,x,v)\right|dtdxdv\\ &\quad + \int_Q |f_h(0,x,v) - f_0(x,v)|\;|\varphi(0,x,v)|dxdv,\end{aligned}$$

with the discrete initial data defined, for example, by

$$f_h(0,x,v) = \frac{1}{|C_{i,j}|}\int_{C_{i,j}} f_0(x,v)dxdv \quad \forall (x,v)\in C_{i,j}.$$

Using the assumption on the initial data $f_0 \in L^1(Q) \cap L^\infty(Q)$, we then have

$$\lim_{h\to 0} \int_Q |f_h(0,x,v) - f_0(x,v)| \, |\varphi(0,x,v)| dx dv = 0.$$

Moreover, from the inequality on the term EF_{2h} given by (30) in Lemma 3.3, we have

$$\sum_{n,i,j} |f_{i,j}^{n+1} - f_{i,j}^n| \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \left| \frac{\partial \varphi}{\partial t}(t,x,v) \right| dt dx dv \leq C \| \frac{\partial \varphi}{\partial t} \|_{L^\infty} \Delta t^{1/2}.$$

Then,

$$|E_1 + E_{1,0}| \to 0 \quad \text{ as } h \to 0. \tag{31}$$

Now we deal with the terms E_2 and $E_{2,0}$. Therefore, we first introduce the notation

$$\begin{aligned} E_{2,1} \; = \; \sum_{n,i,j} \Bigg[& v_j^+ \, (f_{i,j}^n - f_{i-1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i-1/2}, v) dv dt \\ & + v_j^- \, (f_{i,j}^n - f_{i+1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v) dv dt \\ & + E_i^{n+} \, (f_{i,j}^n - f_{i,j-1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j-1/2}) dx dt \\ & + E_i^{n-} \, (f_{i,j}^n - f_{i,j+1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j+1/2}) dx dt \Bigg]. \end{aligned}$$

On the one hand, we compare E_2 and $E_{2,1}$:

$$\begin{aligned} |E_2 - E_{2,1}| \; = \; \Bigg| \sum_{n,i,j} \Bigg[& v_j^+ \, (f_{i,j}^n - f_{i-1,j}^n) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x_{i-1/2}, v) dv dt \right] \\ & + v_j^- \, (f_{i,j}^n - f_{i+1,j}^n) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x_{i+1/2}, v) dv dt \right] \\ & + E_i^{n+} \, (f_{i,j}^n - f_{i,j-1}^n) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x, v_{j-1/2}) dx dt \right] \\ & + E_i^{n-} \, (f_{i,j}^n - f_{i,j+1}^n) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x, v_{j-1/2}) dx dt \right] \Bigg] \Bigg|. \end{aligned}$$

Using the inequality on EF_{1h} given by (30) in Lemma 3.3, there exists $c > 0$ depending only on T, R, L, f_0, α, ξ such that the following inequality holds:

$$|E_2 - E_{2,1}| \leq c \, \|\nabla_{(x,v)} \varphi\|_{L^\infty} \; h^{1/2}. \tag{32}$$

On the other hand, we estimate $|E_{2,0} + E_{2,1}|$, rewriting the term $E_{2,1}$ and using the boundary conditions, it yields the following (we remind that φ is compactly supported in velocity):

$$\begin{aligned} E_{2,1} \; = \; & -\sum_{n,i,j} f_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} v_j \, \frac{\partial \varphi}{\partial x}(t,x,v) \, + \, E_i^n \, \frac{\partial \varphi}{\partial v}(t,x,v) dv dx dt \\ & + \sum_{n,j} v_j^+ \, g_j^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, 0, v) dv dt. \end{aligned}$$

Therefore,

$$
\begin{aligned}
|E_{2,0}+E_{2,1}| &\leq \sum_{n,i,j} f_{i,j}^n \left[\int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \left|v-v_j\right| \left|\frac{\partial\varphi}{\partial x}(t,x,v)\right| + \left|E_h(t,x)-E_i^n\right| \left|\frac{\partial\varphi}{\partial v}(t,x,v)\right| dxdvdt\right] \\
&\leq \|\nabla\varphi\|_{L^\infty} \sum_{n,i,j} \Delta t\, \Delta x_i\, \Delta v_j\, f_{i,j}^n\, [\Delta v_j + \sup|E_h(t,x) - E_i^n|]
\end{aligned}
$$

and there exists $C>0$, only depending on T, R, L, f_0, α, ξ, such that

$$
|E_{2,0}+E_{2,1}| \leq C\, \|\nabla\varphi\|_{L^\infty}\, h. \tag{33}
$$

It remains to estimate the last term E_3. Using the definition of $s_{i+1/2,j}$ and $s_{i,j+1/2}$ and performing a discrete integration by part, we get

$$
\begin{aligned}
E_3 &= \Delta t \sum_{n,i,j} \Delta v_j \left[v_j^+\, \sigma_{i-1/2,j}\,(f_{i,j}^n - f_{i-1,j}^n) - v_j^-\, \sigma_{i+1/2,j}\,(f_{i,j}^n - f_{i+1,j}^n)\right] [\varphi_{i-1,j}^n - \varphi_{i,j}^n] + \\
&\qquad \Delta x_i \left[E_i^{n+}\, \sigma_{i,j-1/2}\,(f_{i,j}^n - f_{i,j-1}^n) - E_i^{n-}\, \sigma_{i,j+1/2}\,(f_{i,j}^n - f_{i,j+1}^n)\right] [\varphi_{i,j-1}^n - \varphi_{i,j}^n].
\end{aligned}
$$

However, we know that

$$
|\varphi_{i-1,j}^n - \varphi_{i,j}^n| \leq \|\frac{\partial\varphi}{\partial x}\|_{L^\infty} \left(\frac{\Delta x_{i-1}+\Delta x_i}{2}\right).
$$

and using the estimate on EF_{1H} in Lemma 3.3, it yields there exists a constant $C>0$ such that

$$
|E_3| \leq C\, \|\nabla_{(x,v)}\varphi\|_{L^\infty}\, h^{1/2}. \tag{34}
$$

Finally, recalling that $E_1 + E_2 + E_3 = 0$, we obtain

$$
\begin{aligned}
\epsilon(\Delta t,h) &= \int_{Q_T} f_h \left(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x} + E_h(t,x)\,\frac{\partial\varphi}{\partial v}\right) dtdxdv + \int_Q f_0(x,v)\varphi(0,x,v)dxdv \\
&= E_{1,0} + E_{2,0} \\
&= E_{1,0} + E_1 + E_{20} + E_{2,1} - E_{2,1} + E_2 + E_3,
\end{aligned}
$$

and from the previous estimates, we proved there exists a constant C depending only on φ, f_0, L, T, α, ξ such that

$$
\begin{aligned}
|E_{1,0} + E_1| &\leq C\,(\|f_0 - f_h(0)\|_{L^1} + \Delta t^{1/2}), \\
|E_{2,0} - E_2| &\leq C\, h^{1/2}, \\
|E_{2,0} + E_{2,1}| &\leq C\, h. \\
|E_3| &\leq C\, h^{1/2}.
\end{aligned}
$$

Then, $\epsilon(\Delta t,h) \to 0$ as $h \to 0$.

As we know

$$
f_h(t,x,v) \rightharpoonup f(t,x,v) \text{ in } L^\infty(Q_T) \text{ weak-}\star
$$

and

$$
E_h(t,x) \to E(t,x) \text{ in } C^0(\overline{\Omega}_T),
$$

we have shown that the limit pair (f,E) of a subsequence $(f_h,E_h)_{h>0}$ is a solution of the Vlasov equation (1). To conclude, we have to prove that this couple is also a solution of the Poisson equation.

Remark 4.1. *In practical calculation, we use a large but finite bound M for the velocity space. In this paper, we assume that as $h \to 0$, the support of the velocity space goes to infinity, and the stability condition (20) imposes on us that*

$$
\exists\, \varepsilon \in (0,1), \quad v_h \simeq \frac{1}{h^\epsilon}, \quad \textit{and} \quad \Delta t \simeq \frac{h^2}{h^{1-\epsilon}+h} \simeq h^{1+\epsilon}.
$$

4.3. **Convergence to the solution of the Poisson equation.** We have already proved that there exists a subsequence of $(E_h)_{h>0}$ and $E \in C^0(\overline{\Omega}_T)$ such that E_h converges to E and $\|E_h - \tilde{E}_h\|_{L^\infty}$ goes to zero when h goes to zero. Hence, we know that up to a sub-sequence $\tilde{E}_h$ converges to E. Now let us prove that E is solution to the Poisson equation.

On the one hand, for all test functions which belong to $C^1_c([0,T) \times (0,L))$, we set φ_i^n such that

$$\varphi_i^n := \frac{1}{\Delta t\, \Delta x_i} \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x) dx dt$$

and multiply the finite volume scheme (15) by $\Delta t\, \Delta x_i\, \varphi_i^n$, sum over $i \in \{0, \ldots, n_x - 1\}$ and $n \in \{0, \ldots, N_T = \frac{T}{\Delta t}\}$, it gives $T_1 + T_2 = 0$ with

$$\begin{aligned} T_1 &= \Delta t \sum_{i,n} \Delta x_i\, E_i^n \left(\varphi_i^n - \varphi_{i-1}^n\right) \\ T_2 &= \sum_{i,n} \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i^n\, \varphi(t,x) dx dt. \end{aligned}$$

We also set $T_{1,0}$ and $T_{2,0}$

$$\begin{aligned} T_{1,0} &= \int_{\Omega_T} \tilde{E}_h(t,x)\, \frac{\partial \varphi}{\partial x}(t,x)\, dx dt \\ T_{2,0} &= \int_{\Omega_T} \rho_h(t,x)\, \varphi(t,x)\, dt dx, \end{aligned}$$

and observe that $T_2 = T_{2,0}$ and

$$\begin{aligned} |T_1 - T_{1,0}| &= \left| \sum_{i,n} \Delta x_i\, E_i^n \int_{t^n}^{t^{n+1}} \left[\varphi_i^n - \varphi(t, x_{i+1/2}) - \varphi_{i-1}^n + \varphi(t, x_{i+1/2})\right] dt \right| \\ &\leq C_T \, \|\frac{\partial \varphi}{\partial x}\|_{L^\infty}\, h. \end{aligned}$$

The weak formulation infers that the solution of the Vlasov-Poisson system belongs to $C^0([0,T[; \mathcal{D}')$, but observing the electric field E is bounded in $W^{1,\infty}(\Omega_T)$ and the initial data are continuous, we see that the distribution function f is also continuous in (x,v). Let us recall that under our hypothesis, the solution of the Vlasov-Poisson system (1)-(5) is unique; then any subsequence that we considered converges to the same limit and the sequence $(f_h, E_h)_{h>0}$ converges to the unique solution.

5. Numerical Simulations

In this section, we consider the two component Vlasov-Poisson system. Let f_α be the distribution function of species $\alpha \in \{e, i\}$; it satisfies the Vlasov equation

$$\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha}\, E(t,x)\, \frac{\partial f_\alpha}{\partial v} = 0 \tag{35}$$

coupled with the Poisson equation

$$E(t,x) = -\nabla_x \phi(t,x), \quad -\frac{\partial^2 \phi}{\partial x^2}(t,x) = \frac{\rho}{\epsilon_0}, \tag{36}$$

where

$$\rho(t,x) = \sum_{\alpha \in \{i,e\}} \rho_\alpha$$

and

$$\rho_\alpha = q_\alpha \int_{\mathbb{R}} f_\alpha(t,x,v)\,dv, \quad \alpha \in \{e,i\}.$$

In the previous analysis we presented for clarity reasons the single component Vlasov-Poisson system, but the result remains true for the multi-component case. We assume here that $q_e = -q_i = 1$ and $\epsilon_0 = 1$ and $m_e/m_i = 0.001$, which means that ions are more heavy than electrons. We perform numerical simulations for this model with a zero initial datum $f_0 = 0$, $\lambda(t) = 1$ and

$$g(t,v) = \frac{1}{\sqrt{2\,\pi}}\,\exp(-v^2/2).$$

In order to improve time discretization accuracy, the procedure is achieved by a second order Runge-Kutta scheme. We performed numerical simulations for different meshes and only report the results of a simulation using a number of cells $n_x = 128$ in the x-direction, and $n_v = 128$ in the v-direction with $v_{max} = 6$, and the time step $\Delta t = 0.01$ for the conservative finite volume scheme. For these configurations, numerical results are no more sensitive to the mesh and are comparable in term of accuracy. The evolution obtained by the finite volume scheme clearly appears to give a good approximation with 128×128 points. Here, nonlinear effects are so important that it is necessary to control spurious oscillation; the second order scheme is conservative and also preseves positivity of the numerical solution. Moreover, the use of slope correctors in the finite volume scheme allows to damp spurious osccillations. For the distribution function in the (x,v) space, some filaments become smaller than the phase space grid size. Nevertheless, this smooth approximation seems to give a good description of macroscopic values (physics quantities obtained by the integration of moments of the distribution function with respect to v). Indeed, the evolution of the electric energy is still accurate using the second order accuracy.

The processes that are at stake here are highly nonlinear and present discontinuities in phase space. They consist in the excitation of a plasma wave by injected electrons. As the beam progresses in plasma, the amplitude of the plasma wave grows and more electrons are trapped in this wave as shown in Fig. 1. At the same time, the plasma electron are ejected through the right side of the simulation box to neutralize the injected charge with electron beam. In Fig. 2, the modulations of electron density are the result of large plasma frequency oscillations. An increase of either the external potential or the simulation time would have resulted in another regime, related to breaking of the plasma wave, where droplets of accelerated particles are generated in phase space.

Acknowledgement. This work is partially supported by the programm “ANR Jeunes Chercheurs” JCJC-0136 (projet *MNEC*) from the french Ministery of Research.

References

[1] T.D. Arber and R.G.L. Vann, *A critical comparison of Eulerian-grid-based Vlasov solvers*, J. Comput. Physics, **180**, (2002) pp. 339–357.

[2] N. Besse, *Convergence of a semi-Lagrangian scheme for the one-dimensional Vlasov-Poisson system* SIAM J. Numer. Anal. **42** (2004) pp. 350–382.

[3] N. Besse and M. Mehrenberger, *Convergence of classes of high-order semi-Lagrangian schemes for the Vlasov-Poisson system*, Math. Comp. **77** (2008), pp. 93–123

[4] J. P. Boris and D. L. Book, *Solution of continuity equations by the method of flux-corrected transport,* J. Comput. Phys. **20** (1976) pp. 397–431.

[5] J. A. Carrillo and F. Vecil, *Nonoscillatory interpolation methods applied to Vlasov-based models,* SIAM J. Sci. Comput. **29** (2007) pp. 1179–1206.
C. Z. Cheng and G. Knorr, *The integration of the Vlasov equation in configuration space,* J. Comput. Phys., **22** (1976) pp. 330–351.

[6] C. Z. Cheng and G. Knorr, *The integration of the Vlasov equation in configuration space,* J. Comput. Phys., **22** (1976) pp. 330–351.

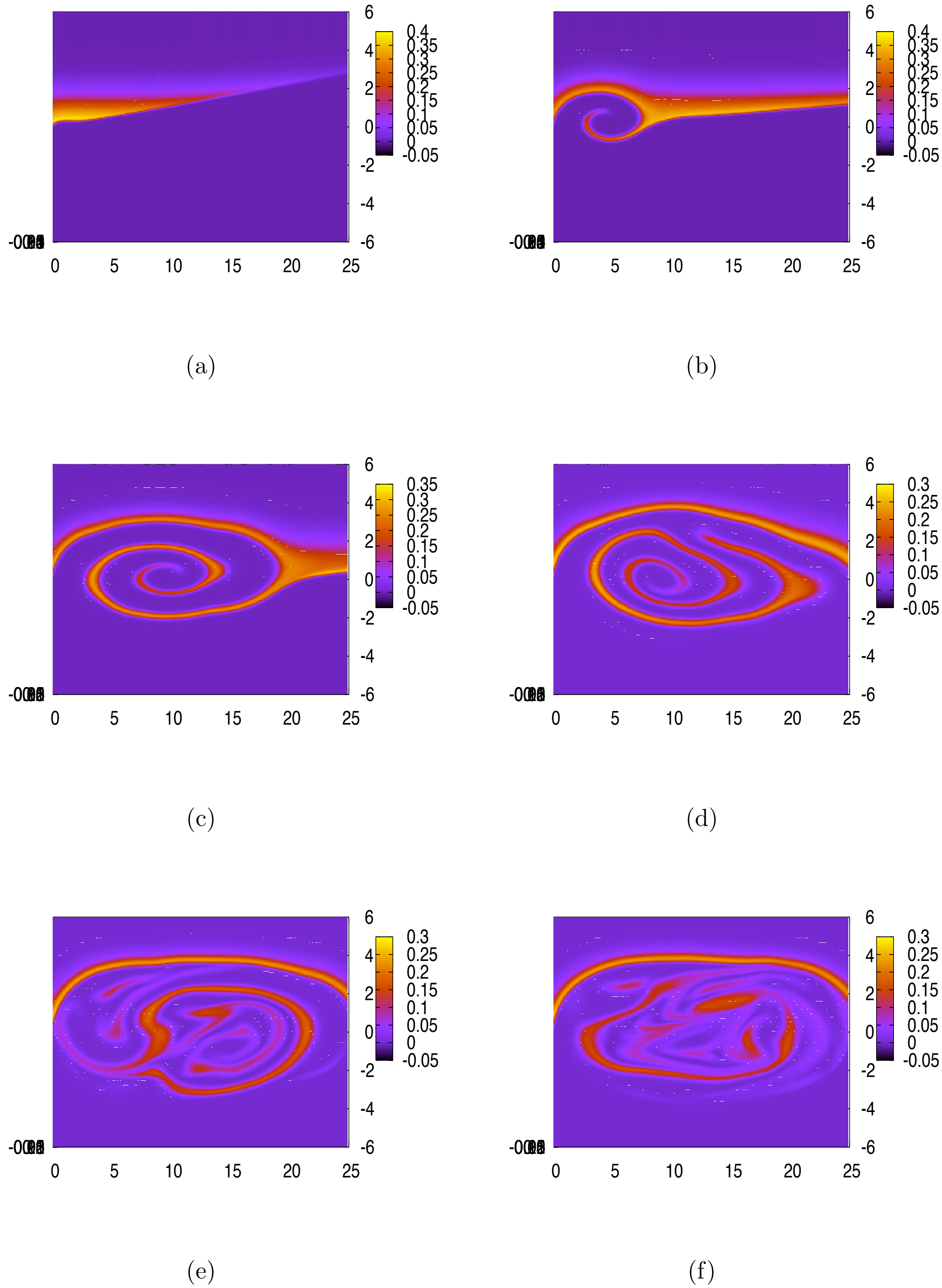

FIGURE 1. *Formation of a phase space vortex : the electron's distribution function* $f_h(t, x, v)$ *at time* t = 10, 30, 50, 80, 120 *and* 150 *obtained with the second order finite volume scheme with* 128×128 *points.*

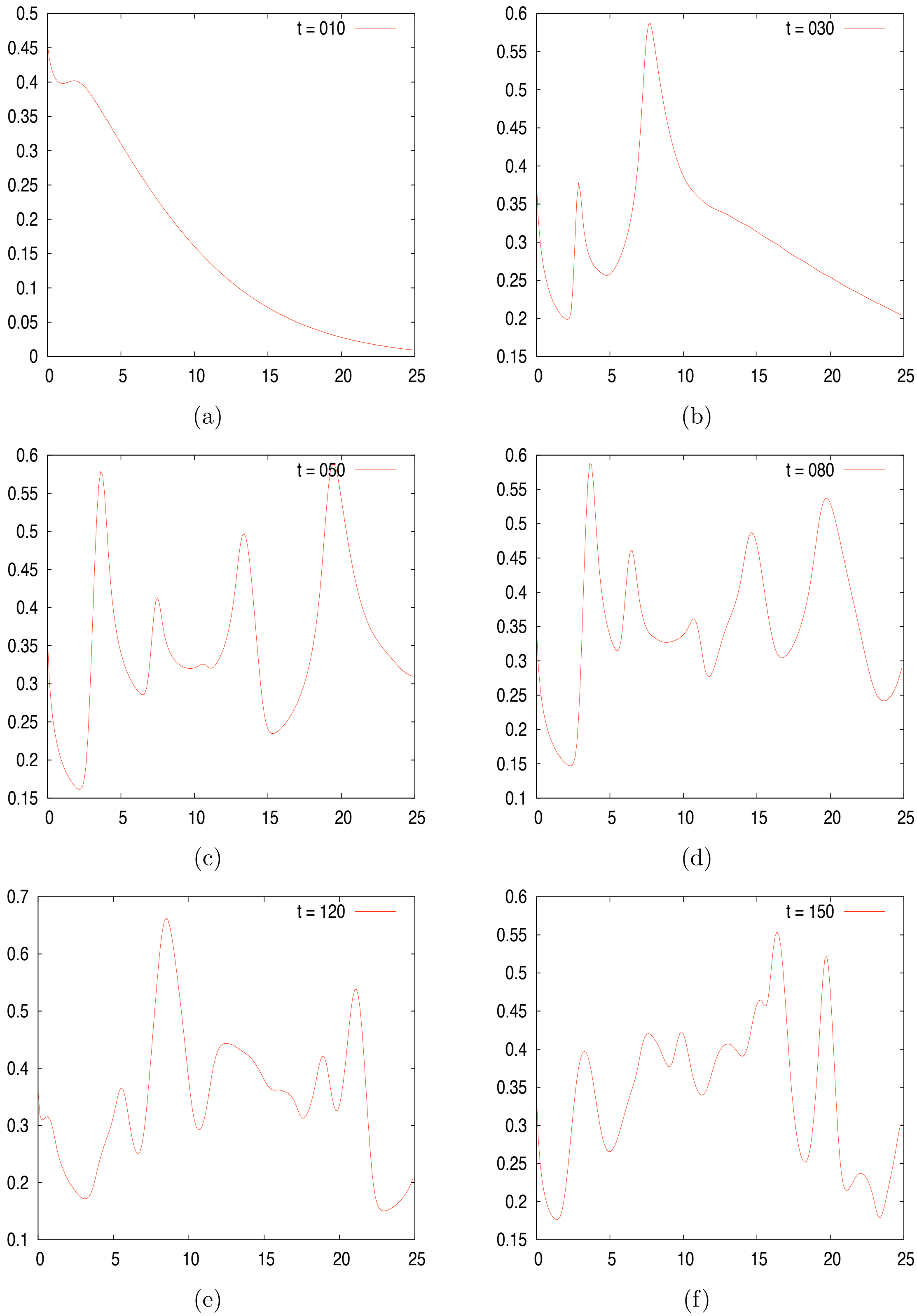

FIGURE 2. *Formation of a phase space vortex : the electron density* $n(t,x)$ *at time* $t = 10$, 30, 50, 80, 120 *and* 150 *obtained with the second order finite volume scheme with* 128×128 *points.*

[7] J. Cooper and A.J. Klimas, *Boundary value problems for the Vlasov-Maxwell equation in one dimension*, J. Math. Anal. Appl. **75** (1980) pp. 306–329.

[8] P. Degond, P.-A. Raviart, *An asymptotic analysis of the Vlasov-Poisson system: the Child-Langmuir law*, Asymptotic Anal. **4** (1991) pp. 187–214.

[9] R. J. DiPerna and P. L. Lions, *Solutions globales d'équations du type Vlasov-Poisson,* C. R. Acad. Sci. Paris Série I Math., **307** (1988) pp. 306–329.

[10] F. Filbet, *Convergence d'un schéma de type volumes finis pour le système de Vlasov-Poisson*, CRAS Paris **330**, série I Mathématique pp. 979–984 (2000).

[11] F. Filbet, *Convergence of a finite volume scheme for the Vlasov-Poisson system*, SIAM J. Numer. Anal. **39** (2001) pp. 1146–1169.

[12] F. Filbet, E. Sonnendrücker and P. Bertrand, *Conservative Numerical schemes for the Vlasov equation*, J. Comput. Phys. **172** (2001) pp. 166–187.

[13] F. Filbet and E. Sonnendrücker, *Comparison of Eulerian Vlasov Solvers*, *Computer Physics Communications* **150** (2003) pp. 247–266.

[14] F. Filbet, Y. Guo and C.-W. Shu, *Analysis of the Reduced Vlasov-Maxwell Model with Boundary Conditions*, Quarterly Applied Math., **63** (2005) pp. 691–714.

[15] C. Greengard, P.-A. Raviart, *A boundary-value problem for the stationary Vlasov-Poisson equation : the plane diode*, Comm. Pure. Appl. Math. **43** (1990) pp. 472–507.

[16] Y. Guo, *Global weak solutions of the Vlasov-Maxwell system with boundary conditions*, Comm. Math. Phys. **154** (1993) pp. 254–263.

[17] Y. Guo, C.W Shu and T. Zhou, *The Dynamics of a Plane Diode*, SIAM J. Math. Anal. **35** (2004) pp. 1617–1635.

[18] A. J. Klimas and W. M. Farrell, *A splitting algorithm for Vlasov simulation with filamentation filtration*, J. Comput. Phys. **110** (1994) pp. 150–163.

[19] F. Poupaud, *Boundary value problems for the stationary Vlasov-Maxwell system*, Forum Math. **4** (1992) pp. 499–527.

[20] J. Schaeffer, *Convergence of a difference scheme for the Vlasov–Poisson–Fokker–Planck system in one dimension,* SIAM J. Numer. Anal. **35** (1997) pp. 1149–1175.

[21] M. Shoucri and G. Knorr, *Numerical integration of the Vlasov equation,* J. Comput. Phys. **14** (1974) pp. 84–92.

[22] Y. Zheng and A. Majda, *Existence of global weak solutions to one-component Vlasov-Poisson and Fokker-Planck-Poisson systems in one space dimension with measures as initial data,* Comm. Pure Appl. Math. **47** (1994) pp. 1365–1401.

[23] T. Nakamura and T. Yabe, *Cubic interpolated propagation scheme for solving the hyper-dimensional Vlasov-Poisson equation in phase space*, Comput. Phys. Communications **120** (1999) pp. 122–154.